AGENDA

1. Preference Learning Tasks (Eyke)
2. **Loss Functions** (Johannes)
   - **a.** **Evaluation of Rankings**
   - b. Weighted Measures
   - c. Evaluation of Bipartite Rankings
   - d. Evaluation of Partial Rankings
3. Preference Learning Techniques (Eyke)
4. Complexity (Johannes)
5. Conclusions

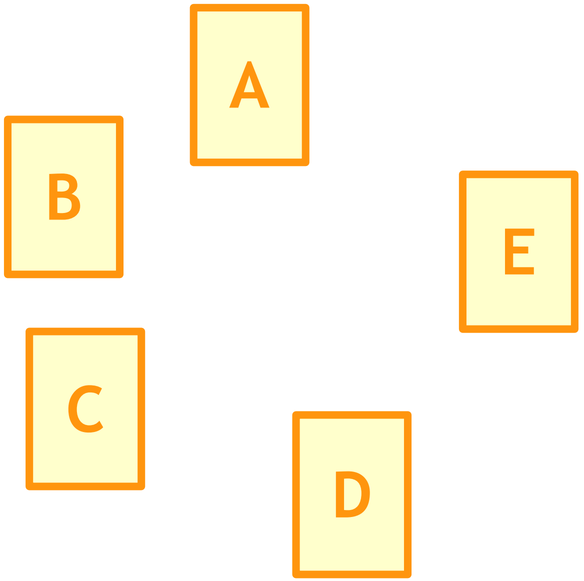
Rank Evaluation Measures

- In the following, we do not discriminate between different ranking scenarios
  - we use the term items for both, objects and labels
- All measures are applicable to both scenarii
  - sometimes have different names according to context
- Label Ranking
  - measure is applied to the ranking of the labels of each examples
  - averaged over all examples
- Object Ranking
  - measure is applied to the ranking of a set of objects
  - we may need to average over different sets of objects which have disjoint preference graphs
    - e.g. different sets of query / answer set pairs in information retrieval

Ranking Errors

- Given:
  - a set of items $X = \{x_1, \ldots, x_c\}$ to rank
    - Example:
      $X = \{A, B, C, D, E\}$

items can be objects or labels



Ranking Errors

- Given:
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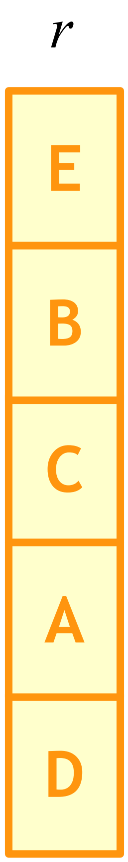
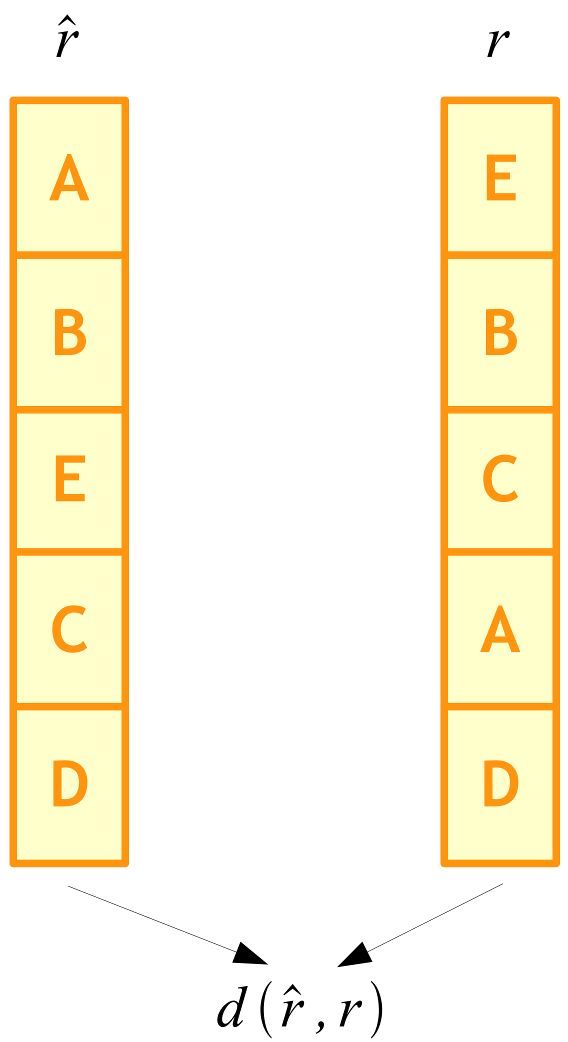
    - *Example:*
      X = {A, B, C, D, E}
  - a target ranking r
    - *Example*:
      E $\succ$ B $\succ$ C $\succ$ A $\succ$ D



Ranking Errors

- Given:
  - a set of items $X = \{x_1, \ldots, x_c\}$ to rank
    - *Example:*
      X = {A, B, C, D, E}
  - a target ranking r
    - *Example*:
      E $\succ$ B $\succ$ C $\succ$ A $\succ$ D
  - a predicted ranking $\hat{r}$
    - *Example*:
      A $\succ$ B $\succ$ E $\succ$ C $\succ$ D
- Compute:
  - a value $d(r, \hat{r})$ that measures the *distance* between the two rankings



Notation

- r and $\hat{r}$ are functions from $X \rightarrow \mathbb{N}$
  - returning the rank of an item x

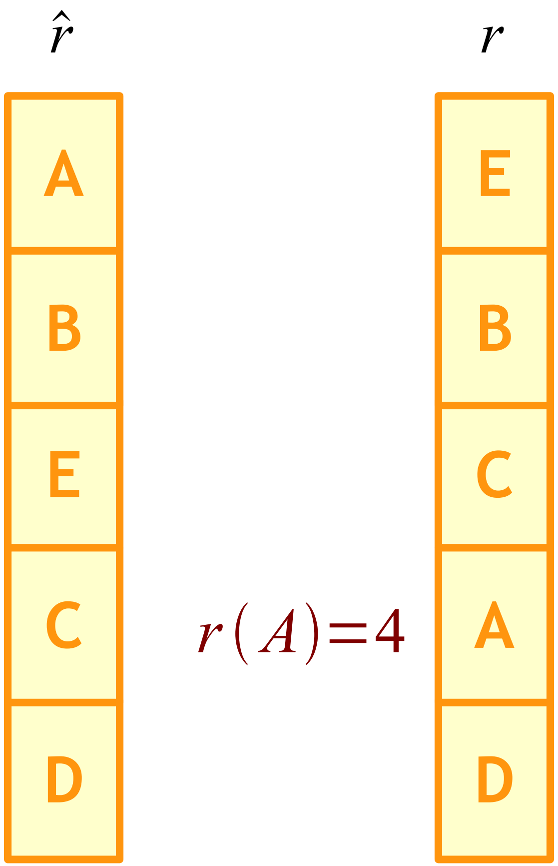
$$\hat{r}(A) = 1$$

- the inverse functions r^{-1}: $\mathbb{N} \rightarrow X$
  - return the item at a certain position

$$\hat{r}^{-1}(1) = A \qquad r^{-1}(4) = A$$

- as a short-hand for $r \circ \hat{r}^{-1}$, we also define function R: $\mathbb{N} \rightarrow \mathbb{N}$
  - $R(i)$ returns the true rank of the i-th item in the predicted ranking

$$R(1) = r(\hat{r}^{-1}(1)) = 4$$



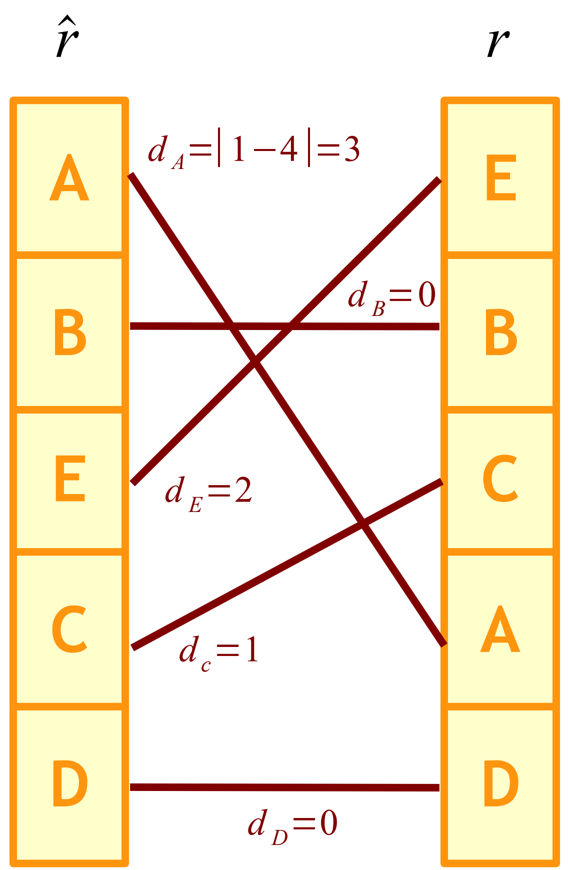
$$r(A) = 4$$

Spearman's Footrule

- Key idea:
  - Measure the sum of absolute differences between ranks

$$D_{SF}(r,\hat{r}) = \sum_{i=1}^{c} |r(x_i) - \hat{r}(x_i)| = \sum_{i=1}^{c} |i - R(i)|$$

$$= \sum_{i=1}^{c} d_{x_i}(r,\hat{r})$$



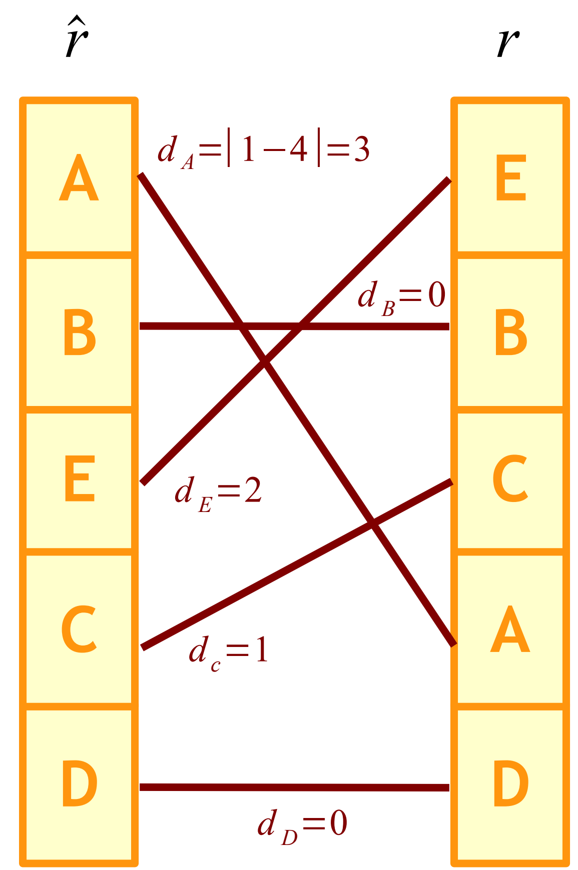
$$\sum_{x_i} d_{x_i} = 3+0+1+0+2 = 6$$

Spearman Distance

- Key idea:
  - Measure the sum of ~~absolute~~ squared differences between ranks

$$D_S(r,\hat{r}) = \sum_{i=1}^{c} (r(x_i) - \hat{r}(x_i))^2 = \sum_{i=1}^{c} (i - R(i))^2$$

$$= \sum_{i=1}^{c} d_{x_i}(r,\hat{r})^2$$



$$\sum_{x_i} d_{x_i}^2 = 3^2 + 0 + 1^2 + 0 + 2^2 = 14$$

- Value range:

$$\min D_S(r,\hat{r}) = 0$$

$$\max D_S(r,\hat{r}) = \sum_{i=1}^{c} ((c-i)-i)^2 = \frac{c\cdot(c^2-1)}{3}$$

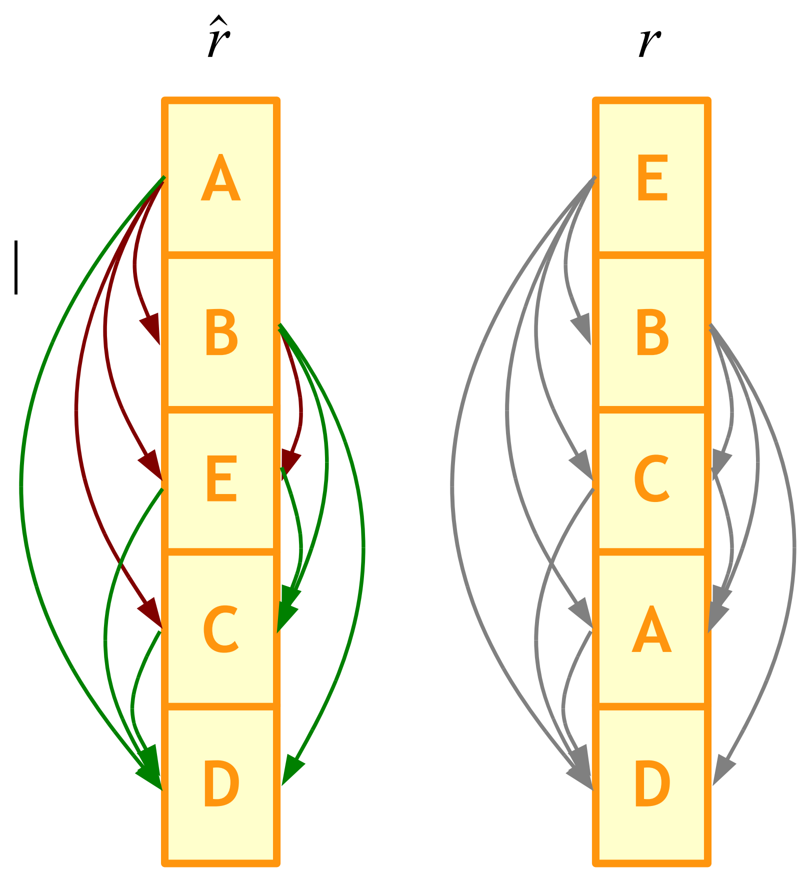
→ Spearman Rank Correlation Coefficient

$$1 - \frac{6\cdot D_S(r,\hat{r})}{c\cdot(c^2-1)} \in [-1,+1]$$

Kendall's Distance

- Key idea:
  - number of item pairs that are inverted in the predicted ranking

$$D_\tau(r,\hat{r}) = \left| \left\{ (i,j) \mid r(x_i) < r(x_j) \wedge \hat{r}(x_i) > \hat{r}(x_j) \right\} \right|$$



$D_\tau(r,\hat{r}) = 4$

- Value range:

$$\min D_\tau(r,\hat{r}) = 0$$

$$\max D_\tau(r,\hat{r}) = \frac{c \cdot (c-1)}{2}$$

→ Kendall's tau

$$1 - \frac{4 \cdot D_\tau(r,\hat{r})}{c \cdot (c-1)} \in [-1,+1]$$

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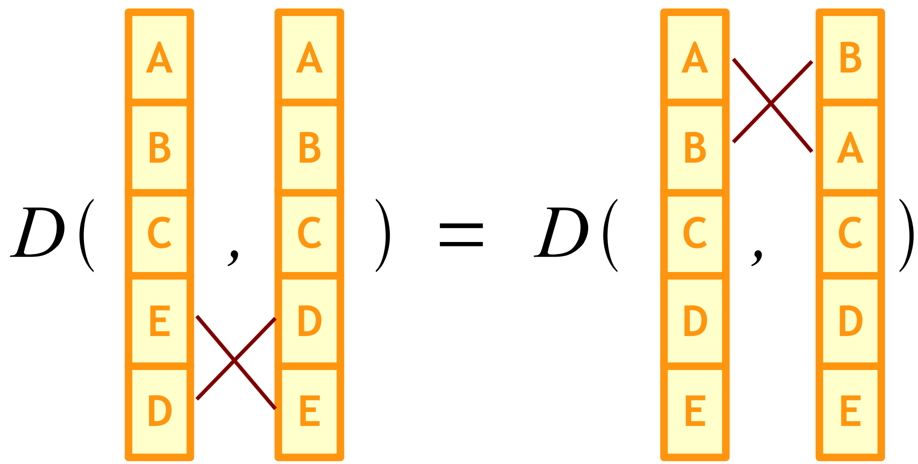
Weighted Ranking Errors

- The previous ranking functions give equal weight to all ranking positions
  - i.e., differences in the first ranking positions have the same effect as differences in the last ranking positions



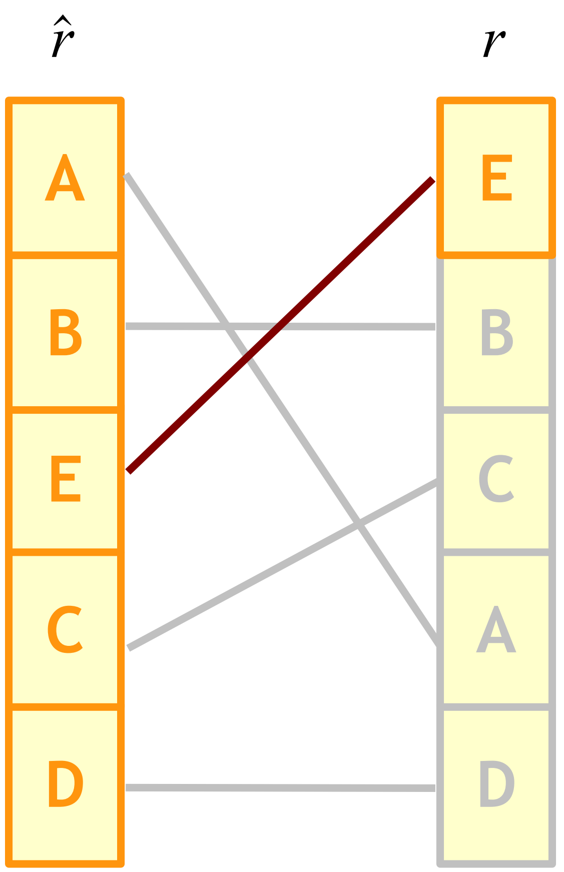
- In many applications this is not desirable
  - ranking of search results
  - ranking of product recommendations
  - ranking of labels for classification
  - ...

$\Rightarrow$ Higher ranking positions should be given more weight

Position Error

- Key idea:
  - in many applications we are interested in providing a ranking where the target item appears a high as possible in the predicted ranking
    - e.g. ranking a set of actions for the next step in a plan
  - Error is the number of wrong items that are predicted before the target item

$$D_{PE}(r, \hat{r}) = \hat{r}(\arg\min_{x \in X} r(x)) - 1$$



$$D_{PE}(r, \hat{r}) = 2$$

- Note:
  - equivalent to Spearman's footrule with all non-target weights set to 0

$$D_{PE}(r, \hat{r}) = \sum_{i=1}^{c} w_i \cdot d_{x_i}(r, \hat{r})$$

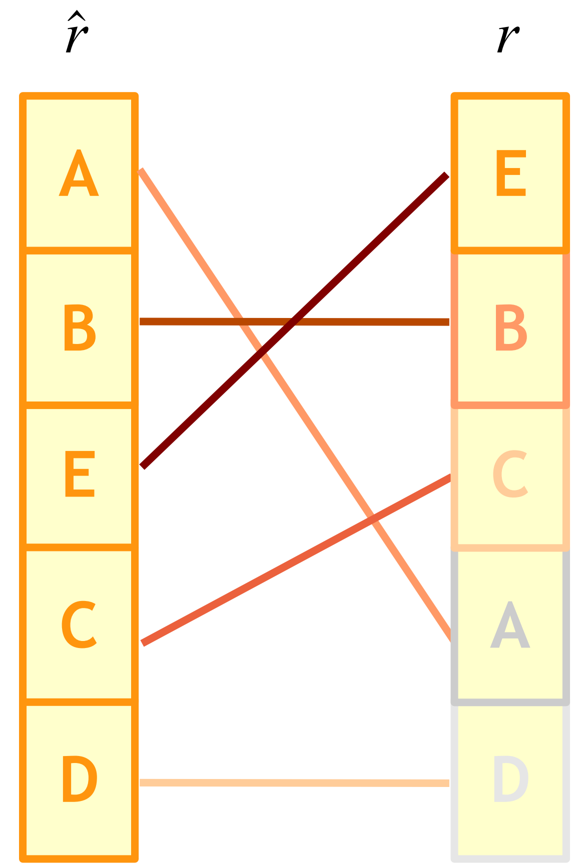
with $w_i = [\![x_i = \arg\min_{x \in X} r(x)]\!]$

Discounted Error

- Higher ranks in the target position get a higher weight than lower ranks

$$D_{DR}(r,\hat{r}) = \sum_{i=1}^{c} w_i \cdot d_{x_i}(r,\hat{r})$$

with $w_i = \dfrac{1}{\log(r(x_i)+1)}$



$$D_{DR}(r,\hat{r}) = \frac{3}{\log 2} + 0 + \frac{1}{\log 4} + 0 + \frac{2}{\log 6}$$

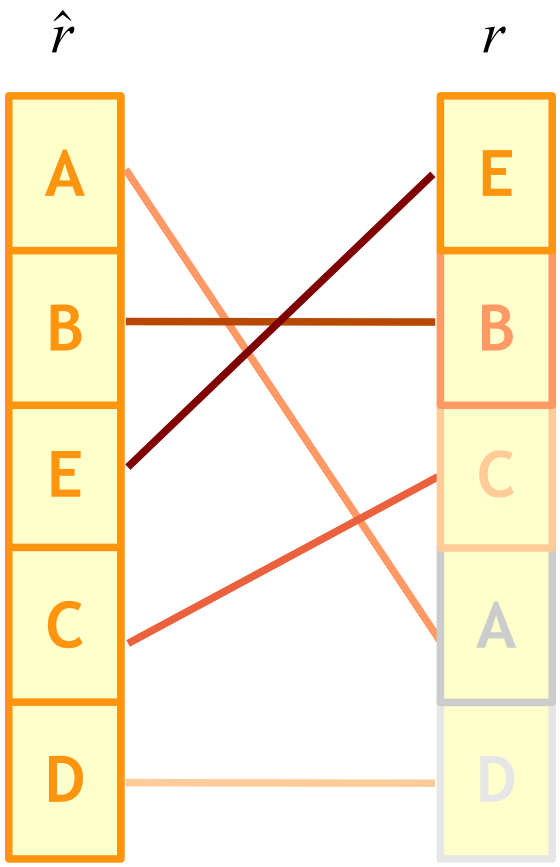
(Normalized) Discounted Cumulative Gain

- a "positive" version of discounted error:
  Discounted Cumulative Gain (DCG)

$$DCG(r,\hat{r}) = \sum_{i=1}^{c} \frac{c - R(i)}{\log(i+1)}$$

- Maximum possible value:
  - the predicted ranking is correct,
    i.e. $\forall i : i = R(i)$
  - Ideal Discounted Cumulative Gain (IDCG)

$$IDCG = \sum_{i=1}^{c} \frac{c - i}{\log(i+1)}$$



- Normalized DCG (NDCG)

$$NDCG(r,\hat{r}) = \frac{DCG(r,\hat{r})}{IDCG}$$

$$NDCG(r,\hat{r}) = \frac{\frac{1}{\log 2} + \frac{3}{\log 3} + \frac{4}{\log 4} + \frac{2}{\log 5} + \frac{0}{\log 6}}{\frac{4}{\log 2} + \frac{3}{\log 3} + \frac{2}{\log 4} + \frac{1}{\log 5} + \frac{0}{\log 6}}$$

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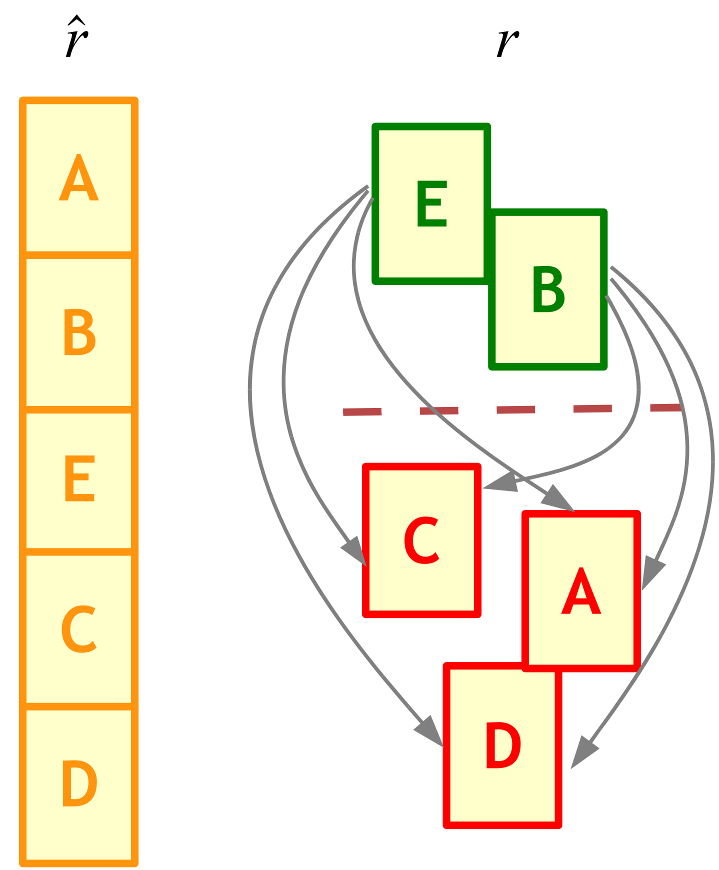
Bipartite Rankings

Bipartite Rankings

- The target ranking is not totally ordered but a *bipartite graph*
- The two partitions may be viewed as preference levels $L = \{0, 1\}$
  - all c_1 items of level 1 are preferred over all c_0 items of level 0
- We now have fewer preferences
  - for a total order: $\frac{c}{2} \cdot (c-1)$
  - for a bipartite graph: $c_1 \cdot (c - c_1)$

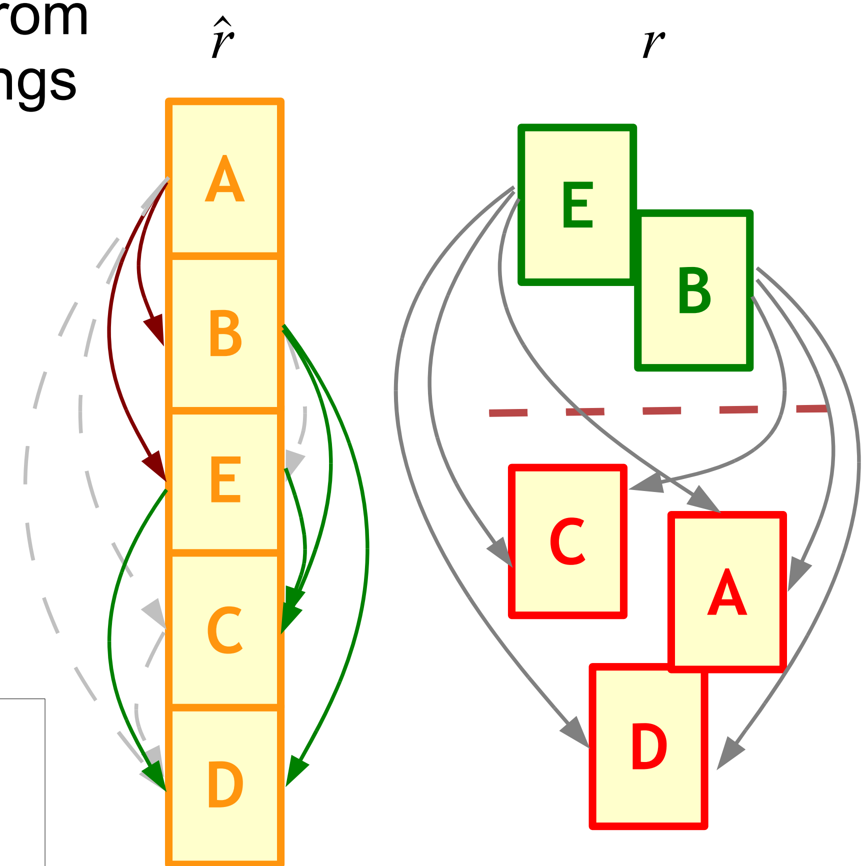


Evaluating Partial Target Rankings

- Many Measures can be directly adapted from total target rankings to partial target rankings
- Recall: Kendall's distance
  - number of item pairs that are inverted in the target ranking

    $$D_\tau(r,\hat{r}) = \left| \left\{ (i,j) \mid r(x_i) < r(x_j) \wedge \hat{r}(x_i) > \hat{r}(x_j) \right\} \right|$$

  - can be directly used
  - in case of normalization, we have to consider that fewer items satisfy $r(x_i) < r(x_j)$
- Area under the ROC curve (AUC)
  - the AUC is the fraction of pairs of (p,n) for which the predicted score $s(p) > s(n)$
    - Mann Whithney statistic is the absolute number
  - This is 1 - normalized Kendall's distance for a bipartite preference graph with $L = \{p,n\}$



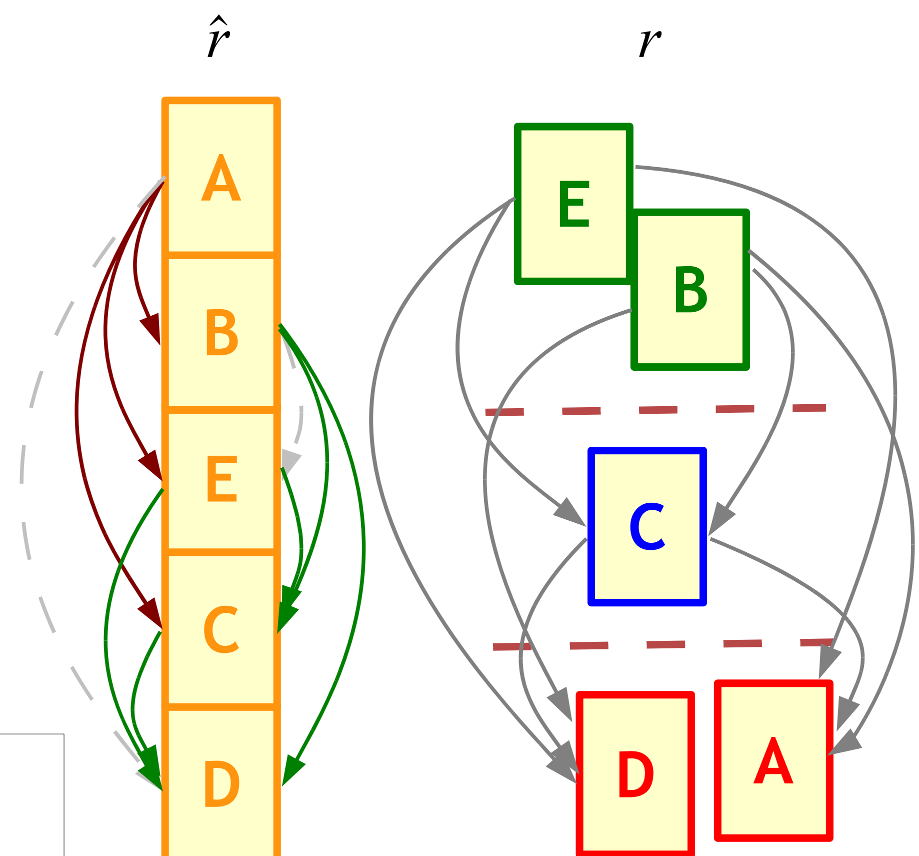
$$D_\tau(r,\hat{r}) = 2$$

$$AUC(r,\hat{r}) = \frac{4}{6}$$

Evaluating Multipartite Rankings

- Multipartite rankings:
  - like Bipartite rankings
  - but the target ranking r consists of *multiple* relevance levels $L = \{1 \ldots l\}$, where $l < c$
  - total ranking is a special case where each level has exactly one item
- # of preferences $= \sum_{(i,j)} c_i \cdot c_j \leq \frac{c^2}{2} \cdot (1 - \frac{1}{l})$
  - c_i is the number of items in level I
- C-Index [Gnen & Heller, 2005]
  - straight-forward generalization of AUC
  - fraction of pairs (x_i, x_j) for which

$$l(i) > l(j) \wedge \hat{r}(x_i) < \hat{r}(x_j)$$



$D_\tau(r, \hat{r}) = 3$

$\text{C-Index}(r, \hat{r}) = \frac{5}{8}$

Evaluating Multipartite Rankings

C-Index

- the C-index can be rewritten as a weighted sum of pairwise AUCs:

$$\text{C-Index}(r,\hat{r}) = \frac{1}{\sum_{i,j>i} c_i \cdot c_j} \sum_{i,j<i} c_i \cdot c_j \cdot \text{AUC}(r_{i,j}, \hat{r}_{i,j})$$

where $r_{i,j}$ and $\hat{r}_{i,j}$ are the rankings r and $\hat{r}$ restricted to levels i and j.

Jonckheere-Terpstra statistic

- is an *unweighted* sum of pairwise AUCs:

$$\text{m-AUC} = \frac{2}{l \cdot (l-1)} \sum_{i,j>i} \text{AUC}(r_{i,j}, \hat{r}_{i,j})$$

- equivalent to well-known multi-class extension of AUC
  [Hand & Till, MLJ 2001]

Note:
C-Index and m-AUC can be optimized by optimization of pairwise AUCs

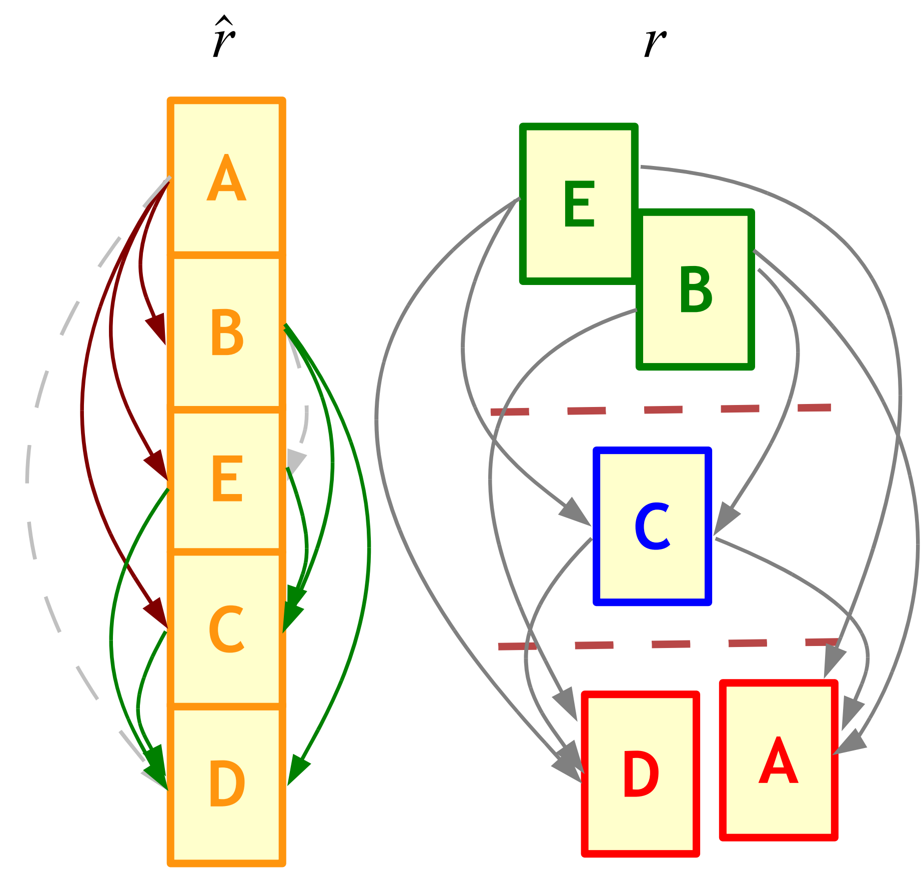
Normalized Discounted Cumulative Gain

[Jarvelin & Kekalainen, 2002]

- The original formulation of (normalized) discounted cumulative gain refers to this setting

$$DCG(r, \hat{r}) = \sum_{i=1}^{c} \frac{l(i)}{\log(i+1)}$$

  - the sum of the true (relevance) levels of the items
  - each item weighted by its rank in the predicted ranking
- Examples:
  - retrieval of relevant or irrelevant pages
    - 2 relevance levels
  - movie recommendation
    - 5 relevance levels



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Evaluating Partial Structures in the Predicted Ranking

- For fixed types of partial structures, we have conventional measures
  - bipartite graphs → binary classification
    - accuracy, recall, precision, F1, etc.
    - can also be used when the items are labels!
      - e.g., accuracy on the set of labels for multilabel classification
  - multipartite graphs → ordinal classification
    - multiclass classification measures (accuracy, error, etc.)
    - regression measures (sum of squared errors, etc.)

- For general partial structures
  - some measures can be directly used on the reduced set of target preferences
    - Kendall's distance, Gamma coefficient
  - we can also use set measures on the set of binary preferences
    - both, the source and the target ranking consist of a set of binary preferences
    - e.g. Jaccard Coefficient
      - size of interesection over size of union of the binary preferences in both sets

Gamma Coefficient

- Key idea: normalized difference between
  - number of correctly ranked pairs (Kendall's distance)

    $d = D_\tau(r, \hat{r})$

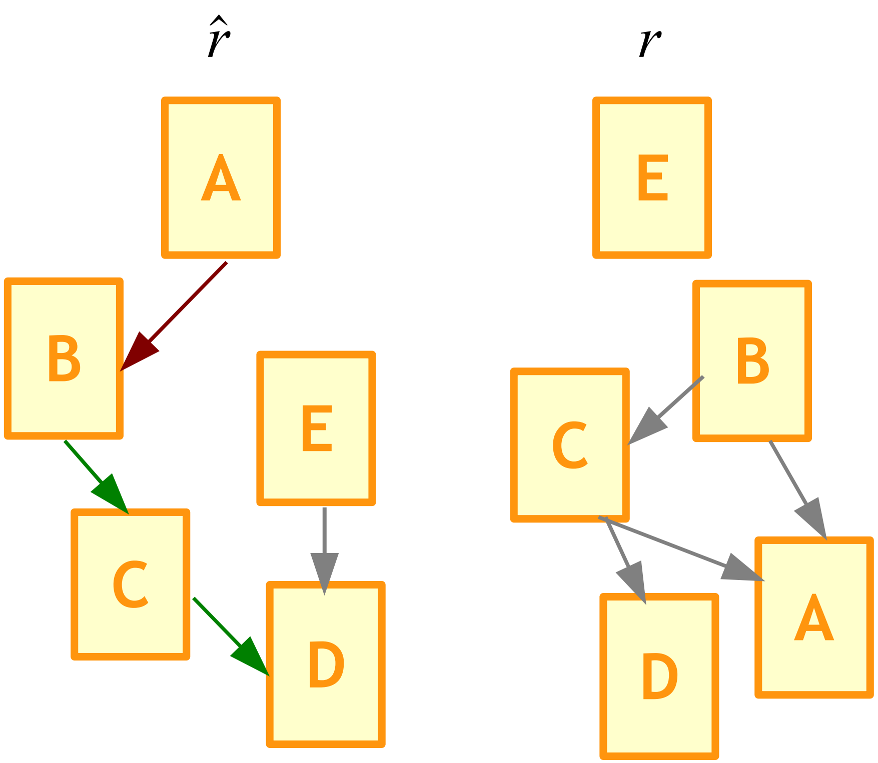
  - number of incorrectly ranked pairs

    $\bar{d} = \left| \left\{ (i, j) \mid r(x_i) < r(x_j) \wedge \hat{r}(x_i) < \hat{r}(x_j) \right\} \right|$

- Gamma Coefficient
  [Goodman & Kruskal, 1979]

  $$\gamma(r, \hat{r}) = \frac{d - \bar{d}}{d + \bar{d}} \in [-1, +1]$$

  - Identical to Kendall's tau if both rankings are total
    - i.e., if $d + \bar{d} = \frac{c \cdot (c-1)}{2}$



$$\gamma(r, \hat{r}) = \frac{2-1}{2+1} = \frac{1}{3}$$

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